

# Global weak solutions to the two-dimensional Navier-Stokes equations of compressible heat-conducting flows with symmetric data and forces

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## Abstract

We prove the global existence of weak solutions to the Navier-Stokes equations of compressible heat-conducting fluids in two spatial dimensions with initial data and external forces which are large and spherically symmetric. The solutions will be obtained as the limit of the approximate solutions in an annular domain. We first derive a number of regularity results on the approximate physical quantities in the “fluid region”, as well as the new uniform integrability of the velocity and temperature in the entire space-time domain by exploiting the theory of the Orlicz spaces. By virtue of these a priori estimates we then argue in a manner similar to that in [Arch. Rational Mech. Anal. **173** (2004), 297-343] to pass to the limit and show that the limiting functions are indeed a weak solution which satisfies the mass and momentum equations in the entire space-time domain in the sense of distributions, and the energy equation in any compact subset of the “fluid region”.

*Keywords:* Global weak solutions, 2D Navier-Stokes equations, heat-conducting flows, spherically symmetric solutions, Orlicz spaces.

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## 1. Introduction

The two-dimensional Navier-Stokes equations of compressible heat-conducting fluids express the conservation of mass, and the balance of momentum and energy, which can be written as follows in Eulerian coordinates.

$$\varrho_t + \operatorname{div}(\varrho \mathbf{u}) = 0, \quad (1.1)$$

$$(\varrho \mathbf{u})_t + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\varrho, \theta) = \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \varrho \mathbf{f}, \quad (1.2)$$

$$\begin{aligned} (\varrho E)_t + \operatorname{div}(\varrho E + P(\varrho, \theta) \mathbf{u}) &= \Delta \left( \kappa \theta + \frac{1}{2} \mu |\mathbf{u}|^2 \right) + \mu \operatorname{div}[(\nabla \mathbf{u}) \mathbf{u}] \\ &\quad + \lambda \operatorname{div}[(\operatorname{div} \mathbf{u}) \mathbf{u}] + \varrho \mathbf{u} \cdot \mathbf{f}. \end{aligned} \quad (1.3)$$

Here  $\varrho, \mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ , and  $\theta$ ,

$$E = \frac{|u|^2}{2} + \theta \quad \text{and} \quad P = K \varrho \theta$$

are the density, velocity, temperature, total energy density and pressure of an ideal gas (with unit specific heat), respectively;  $\mu$  and  $\lambda$  are the constant viscosity coefficients satisfying  $\mu > 0$

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and  $\mu + \lambda \geq 0$ ,  $\kappa > 0$  is the heat-conduction coefficient;  $\mathbf{f} = (f_1, f_2)$  is the external force, and  $\nabla \mathbf{u}$  denotes the gradient of the velocity vector with respect to the spatial variable  $\mathbf{x} \in \mathbb{R}^2$ .

We consider an initial boundary value problem for the system (1.1)–(1.3) in a ball  $\Omega := \{\mathbf{x} \in \mathbb{R}^2; |\mathbf{x}| < R\}$  with boundary conditions

$$\mathbf{u} = \mathbf{0}, \quad \frac{\partial \theta}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega \quad (1.4)$$

and initial conditions

$$(\varrho, \mathbf{u}, \theta)|_{t=0} = (\varrho, \mathbf{u}_0, \theta_0), \quad (1.5)$$

where  $\mathbf{n}$  is the outer normal vector to  $\partial\Omega$ .

In the spherically symmetric case, namely,

$$\varrho(t, \mathbf{x}) = \varrho(t, r), \quad \mathbf{u}(t, \mathbf{x}) = u(t, r) \frac{\mathbf{x}}{r}, \quad \theta(t, \mathbf{x}) = \theta(t, r), \quad \mathbf{f}(t, \mathbf{x}) = f(t, r) \frac{\mathbf{x}}{r}, \quad (1.6)$$

where  $r = r(\mathbf{x}) := |\mathbf{x}|$ , the system (1.1)–(1.3) takes the form:

$$\varrho_t + (\varrho u)_r + \frac{\varrho u}{r} = 0, \quad (1.7)$$

$$(\varrho u)_t + (\varrho u^2)_r + \frac{\varrho u^2}{r} + P_r(\varrho, \theta) - \nu \left( u_r + \frac{u}{r} \right)_r = \varrho f, \quad (1.8)$$

$$(\varrho \theta)_t + (\varrho u \theta)_r + \frac{\varrho u \theta}{r} - \kappa \left( \theta_{rr} + \frac{\theta_r}{r} \right) + P(\varrho, \theta) \left( u_r + \frac{u}{r} \right) = \mathcal{Q}, \quad (1.9)$$

where

$$\nu := \lambda + 2\mu, \quad \mathcal{Q} = \nu \left( u_r + \frac{u}{r} \right)^2 - \frac{2\mu}{r} \partial_r u^2 \geq 0. \quad (1.10)$$

The boundary and initial conditions become

$$u = \theta_r = 0 \quad \text{at } r = R, \quad (1.11)$$

$$(\varrho, u, \theta)|_{t=0} = (\varrho, u_0, \theta_0), \quad 0 < r < R. \quad (1.12)$$

The main purpose of this paper is to prove the global existence of weak solutions to the problem (1.1)–(1.5) when the initial data and external forces are large and spherically symmetric. Our work is motivated by the paper of Hoff and Jenssen [9] where they studied the spherically and cylindrically symmetric nonbarotropic flows with large data and forces, and established the global existence of weak solutions to the compressible nonbarotropic Navier-Stokes equations in the “fluid region”. In the entire space-time domain, however, the momentum equation in [9, Theorem 1.1] only holds weakly with a nonstandard interpretation of the viscosity terms as distributions. A natural question is to ask whether the momentum equation holds in the standard sense of distributions. A positive answer was given recently by Zhang, Jiang and Xie [17] for a screw pinch model arisen from plasma physics when the heat conductivity  $\kappa$  satisfies certain growth conditions. In the present paper, based on some new uniform global estimates of  $u$  and  $\theta$  (see Lemma 2.4) which are established by applying the theory of the Orlicz spaces, we can give a positive result for the two-dimensional Navier-Stokes equations of compressible heat-conducting fluids (1.1)–(1.3), improving therefore the result of [9].

We now give the precise statement of our assumptions and results. The external force  $\mathbf{f}$  is assumed to satisfy

$$\mathbf{f} \in L^1((0, T), L^\infty(\Omega)) \cap L^\infty((0, T), L^2(\Omega)) \quad (1.13)$$

for each  $T > 0$ . The initial data  $(\varrho_0, \mathbf{u}_0, \theta_0)$  are assumed to satisfy

$$C_0^{-1} \leq \varrho_0 \leq C_0, \quad C_0^{-1} \leq \theta_0 \text{ a.e. in } \Omega, \quad (1.14)$$

$$\int_{\Omega} \varrho_0 S(\varrho_0, \mathbf{u}_0, \theta_0) d\mathbf{x} \leq C_0 \quad \text{for some positive constant } C_0, \quad (1.15)$$

where  $S$  is the entropy density in the form of

$$S(\varrho, \mathbf{u}, \theta) = K\Psi(\varrho^{-1}) + \Psi(\theta) + \frac{1}{2}|\mathbf{u}|^2 \quad (1.16)$$

with  $\Psi(s) = s - \log s - 1$ . We point out here that there are no smallness or regularity conditions imposed on  $\mathbf{f}$  and  $(\varrho_0, \mathbf{u}_0, \theta_0)$ .

Under the conditions (1.13)–(1.16), we shall prove the following existence theorem on spherically symmetric solutions to the problem (1.1)–(1.5).

**Theorem 1.1.** *Assume that the initial data  $(\varrho_0, \mathbf{u}_0, \theta_0)$  and the external force  $\mathbf{f}$  are spherically symmetric and satisfy the conditions (1.13)–(1.15). Then, the initial boundary problem (1.1)–(1.5) has a global weak solution  $(\varrho, \mathbf{u}, \theta)$  in the form of (1.6) satisfying the following:*

- (a) *The support of  $\varrho$  is bounded on the left by a Hölder curve  $\underline{r}(t) \in C_{\text{loc}}^{0,1/4} : [0, \infty) \rightarrow [0, \infty)$ . Moreover, if  $\mathcal{F}$  is the “fluid region”, defined by*

$$\mathcal{F} := \{(t, \mathbf{x}) \mid t \geq 0 \text{ and } \underline{r}(t) < r(t) \leq R\},$$

*then  $\mathcal{F} \cap \{t > 0\} \cap \{r < R\}$  is open set.*

- (b) *The density  $\varrho \in L_{\text{loc}}^{\infty}(\mathcal{F})$ ,  $\mathbf{u}$  and  $\theta$  are locally Hölder continuous in  $\mathcal{F} \cap \{t > 0\}$ , and the Navier-Stokes equations (1.1)–(1.3) hold in  $\mathcal{D}'(\mathcal{F} \cap \{t > 0\} \cap \{r < R\})$ .*
- (c) *The density  $\varrho \in C([0, \infty), W^{1,\infty}(\Omega)^*)$ . Also,  $\varrho(t, \cdot) \equiv 0$  in  $\Omega \setminus \bar{\mathcal{F}}$ , and if  $\varrho \mathbf{u}$  is taken to be zero in  $\Omega \setminus \bar{\mathcal{F}}$ , then the weak form of the mass equation (1.1) holds for test functions  $\psi \in C^1([t_1, t_2] \times \bar{\Omega})$ :*

$$\int_{\Omega} \varrho \psi d\mathbf{x} \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\Omega} (\varrho \psi_t + \varrho \mathbf{u} \cdot \nabla \psi) d\mathbf{x} dt. \quad (1.17)$$

- (d) *The velocity  $\mathbf{u} \in L_{\text{loc}}^{4/3}([0, \infty), W^{1,4/3}(\Omega))$ . For  $t_1 < t_2$ , if  $\psi \in C^1([t_1, t_2] \times \bar{\Omega})$  vanishes on  $\partial\Omega$ , then, for  $i = 1, 2$ ,*

$$\begin{aligned} & \int_{\Omega} \varrho u_i \psi d\mathbf{x} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\Omega} (\varrho u_i \psi_t + \varrho u_i \mathbf{u} \cdot \nabla \psi + P(\varrho, \theta) \psi_{x_i}) d\mathbf{x} dt \\ &= \int_{t_1}^{t_2} \int_{\Omega} \varrho f_i d\mathbf{x} dt - \int_{t_1}^{t_2} \int_{\Omega} [(\lambda + \mu) \mathbf{u}_{x_i} \cdot \nabla \psi + \mu \nabla u_i \nabla \psi] d\mathbf{x} dt. \end{aligned} \quad (1.18)$$

- (e) *The gradient  $\nabla \theta \in L_{\text{loc}}^1(\mathcal{F})$ , and the weak form of the energy equation (1.3) holds for test functions  $\psi \in C^1([t_1, t_2] \times \bar{\Omega})$  for which there is an  $\eta > 0$  such that  $\text{supp} \psi(t, \cdot) \subset \{\mathbf{x} \mid \underline{r}(t) + \eta \leq r(\mathbf{x})\}$  for each  $t \in [t_1, t_2]$ :*

$$\begin{aligned} & \int_{\Omega} \varrho E \psi d\mathbf{x} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\Omega} (\varrho E \psi_t + (\varrho E + P(\varrho, \theta)) \mathbf{u} \cdot \nabla \psi) d\mathbf{x} dt = \int_{t_1}^{t_2} \int_{\Omega} \varrho \mathbf{f} \cdot \mathbf{u} \psi d\mathbf{x} dt \\ & - \int_{t_1}^{t_2} \int_{\Omega} \left( \kappa \nabla \theta + \frac{1}{2} \mu \nabla |\mathbf{u}|^2 + \mu (\text{div} \mathbf{u}) \mathbf{u} + \lambda (\nabla \mathbf{u}) \mathbf{u} \right) \cdot \nabla \psi d\mathbf{x} dt. \end{aligned} \quad (1.19)$$

- (f) *The total energy energy, minus the mechanical work done by the external force, is weakly nonincreasing in time. That is, if*

$$\mathcal{E}(t) := \int_{\Omega} \varrho(t, \mathbf{x}) \left[ \theta(t, \mathbf{x}) + \frac{1}{2} |\mathbf{u}|^2 \right] d\mathbf{x} dt,$$

then

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_{\Omega} \varrho \mathbf{u} \cdot \mathbf{f} d\mathbf{x} dt - \lim_{b \rightarrow 0} \lim_{j \rightarrow \infty} \int_{\varepsilon_j \leq r(\mathbf{x}) \leq b} (\varrho^j E^j)(t, \mathbf{x}) d\mathbf{x}$$

as a function of  $t$  in  $\mathcal{D}'(0, \infty)$ , where  $E^j = |\mathbf{u}^j|^2/2 + \theta^j$ .

**Remark 1.1.** In [9], Hoff and Jenssen proved that (1.2) holds in  $\mathcal{D}'(\mathcal{F} \cap \{t > 0\} \cap \{r < R\})$ . Here, by exploiting the theory of the Orlicz spaces we are able to derive some new uniform global integrability of the approximate solutions (cf. Lemma 2.4) to show that (1.2) holds in the entire space-time domain in the weak sense (i.e., (d) of Theorem 1.1). Furthermore, this method can be applied to the screw pinch model with positive constant heat-conduction coefficient in [17] and the cylindrically symmetric rotating model of (1.1)–(1.3) (that is, in the symmetric equations (6)–(10) in [9], we take  $u = v$ ,  $w = 0$ ,  $f_1 = f_2$  and  $f_3 = 0$ ) to obtain similar results.

Combining the global a priori estimates derived in Subsection 2.3, we shall prove Theorem 1.1 in Section 4 by the convergence argument similar to that in [9]. For this purpose, we consider the approximate solutions  $(\varrho^j, \mathbf{u}^j, \theta^j)$  of the problem (1.1)–(1.5) in the annular regions  $\Omega^j := \{\mathbf{x} \mid \varepsilon_j < r(\mathbf{x}) < R\}$ , where  $\varepsilon_j$  is a sequence of positive inner radii tending to 0. Since the  $1/r$  singularity in the equations (1.7)–(1.9) plays no role at the stage when  $\varepsilon_j$  is fixed and positive, the global existence of approximate solutions  $(\varrho^j, \mathbf{u}^j, \theta^j)$  for (1.1)–(1.5) can thus be shown in a manner similar to that in [9, 17]. However, to pass to the limit as  $j \rightarrow \infty$  and to show the global existence of weak solutions to the original problem (1.1)–(1.5), we need some  $\varepsilon_j$ -independent a priori estimates. This will be done in Sections 2 and 3. We first prove the global estimates in Section 2, where we derive the standard energy-entropy estimates in Subsection 2.1, and apply these estimates to establish a new uniform integrability of the approximate solutions in the entire spacetime domain by exploiting the theory of Orlicz spaces in Subsections 2.2 and 2.3, which is crucial in the proof of (d) of Theorem 1.1. Then, in Section 3, we list the well-known the pointwise bounds for  $\varrho_j$  and  $\theta_j$  as consequences of the energy and entropy estimates. These pointwise bounds are independent of  $\varepsilon_j$ , but only away from the origin of Lagrangian space. More precisely, as in [9], for any given  $h > 0$  we define the particle position  $r_h^j(t)$  by

$$h = \int_{\varepsilon_j}^{r_h^j(t)} \varrho^j(t, r) r dr$$

from which and the standard energy-entropy estimates it follows that there exists a positive constant  $C(h)$ , depending only on  $h > 0$ , such that  $r_h^j(t) \geq C(h) > 0$ . With this observation, we can obtain that for any fixed  $h > 0$  and  $T > 0$ , there is a positive constant  $C(T, h)$ , depending only on  $h$ ,  $T$  and the initial data, such that

$$C(T, h)^{-1} \leq \varrho^j(t, \mathbf{x}) \leq C(T, h) \quad \text{for any } (t, \mathbf{x}) \in [0, T] \times [r_h^j(t), R].$$

Applying these pointwise bounds, we can get a number of higher-order energy estimates for the approximate solutions in Subsection 3.2, which are also independent of  $\varepsilon_j$  and only away from the origin of Lagrangian space. These  $\varepsilon_j$ -independent bounds enable us to define the “fluid region”  $\mathcal{F}$  (see (a) of Theorem 1.1) and to obtain the uniform Hölder continuity of the quantities on the

compact subsets of  $\mathcal{F} \cap \{t > 0\}$  (see (b) of Theorem 1.1). Finally, all the assertions of (a)–(f) indicated in Theorem 1.1 will be proved in Section 4 by the convergence arguments adapted from Hoff and Jenssen’s paper [9]. We note that the final step of this argument provides a sort of a posteriori validation that the equations (1.7)–(1.9) are indeed the correct forms of the general system (1.1)–(1.3) in the symmetric case considered here.

As pointed out in [9], we still do not have sufficient information to infer that  $\underline{r}(t) \equiv 0$ , nor do we know whether solutions exist for which  $\underline{r} \neq 0$ . The analysis simply shows that  $\underline{r}(t)$  may be positive, and that, if it is, a vacuum state of radius  $\underline{r}(t)$  centered at the origin. In any case, the total mass is conserved in the spherical case, as is clear from (c) of Theorem 1.1, and the total momentum is zero because of symmetry.

We show in (e) only that the energy equation holds on the support of  $\varrho$ , rather than in the entire space-time domain  $(0, \infty) \times \Omega$ . This is partly due to that we cannot obtain higher global regularity of  $\theta$  and  $\mathbf{u}$ . We may regard the restriction in (5) that the test function be supported in  $\mathcal{F}$  as reasonable, since there is no fluid outside  $\mathcal{F}$ , and the model is not really valid there. Additionally, the failure of the analysis to detect whether or not energy is lost ((f) of Theorem 1.1) calls into question the adequacy of the mass, energy, and entropy bounds in Lemma 2.1, which are the only known (global) a priori bounds in the multidimensional case now.

We end this section by mentioning some related existence results for large data in the multidimensional case. The global existence of weak solutions was first shown by Lions [16] for isentropic flows under the assumption that the specific heat ratio  $\gamma > 3n/(n+2)$  where  $n = 2, 3$  denotes the spatial dimension. Then, by using the curl-div lemma to delicately derive certain compactness, and applying Lions’ idea and a technique from [12], Feireisl, Novotný and Petzeltová [4, 6] extended Lions’ existence result to the case  $\gamma > n/2$ . For any  $1 \leq \gamma \leq n/2$ , a global weak solution still exists when the initial data have certain symmetry (e.g., spherical, or axisymmetric symmetry), see [8], [11]–[12]. For non-isentropic flows, the global existence for general data is still not available. Recently, under certain growth conditions upon the pressure, viscosity and heat-conductivity (i.e., radiative gases), Feireisl, et al. obtained the global existence of the so-called “variational solutions” in the sense that the energy equation is replaced by an energy inequality, see [3] for example. However, this result excludes the case of ideal gases unfortunately. The global existence of a solution for large data in the non-isentropic case needs further study.

## 2. Global Estimates

In this section we derive a priori global estimates for any smooth (approximate) solution  $(\varrho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$  of (1.7)–(1.9) together with additional boundary conditions:

$$u^\varepsilon = \theta_r^\varepsilon = 0 \quad \text{at } r = \varepsilon \text{ and } R. \quad (2.1)$$

We assume that the initial data and force are smooth and satisfy the bounds (1.13)–(1.15) with constants independent of  $\varepsilon$  and

$$\int_\varepsilon^R \varrho_0^\varepsilon d\mathbf{x} \equiv M_0 := \int_0^R \varrho_0 d\mathbf{x}.$$

We refer to Section 4 for a brief discussion on the existence of such approximate solutions. As discussed in Section 1, we shall eventually take a sequence of inner radii  $\varepsilon_j \rightarrow 0$  to prove Theorem 1.1. Since  $\varepsilon > 0$  is fixed for the time being, we suppress the dependence on  $j$ .

### 2.1. Energy and Entropy Estimates

We start with the following lemma which states the standard energy and entropy estimates for these approximate solutions.

**Lemma 2.1.** *Let  $(\varrho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$  be a smooth solution of (1.7)–(1.12) defined on  $[0, T] \times [\varepsilon, R]$  with boundary conditions (2.1). Then, there are constants  $M_0$  and  $C(T)$ , such that*

$$\int_{\varepsilon}^R \varrho^\varepsilon(t, r) r dr \equiv M_0, \quad (2.2)$$

$$\int_{\varepsilon}^R (\varrho^\varepsilon E^\varepsilon)(t, r) r dr \leq C(T), \quad (2.3)$$

$$\int_{\varepsilon}^R (\varrho^\varepsilon S^\varepsilon)(t, r) r dr + \int_0^t \int_{\varepsilon}^R \left[ \kappa \left( \frac{\theta_r^\varepsilon}{\theta^\varepsilon} \right)^2 + \frac{\mathcal{Q}}{\theta^\varepsilon} \right] r dr \leq C(T) \quad \text{for all } t \in [0, T], \quad (2.4)$$

where  $E^\varepsilon = (u^\varepsilon)^2/2 + \theta^\varepsilon$ ,  $S^\varepsilon$  is the entropy density defined in (1.16) and  $\mathcal{Q}$  is given in (1.10).

PROOF. The bounds (2.2)–(2.4) are the standard energy estimates which follow directly from the equations (1.7)–(1.9), the boundary conditions and the assumption (1.13) on the external force.  $\square$

## 2.2. Excursion to Theory of the Orlicz Spaces

Before deriving the global estimates on the temperature and velocity, we recall some well-known results concerning the Orlicz spaces (see, for example, [1, 15] for details), which are often used to investigate the 2D compressible Navier-Stokes equations (see [13, 10, 2] for example).

**Definition 2.1** (Young's function). *We say that  $\Phi$  is a Young's (or  $N$ -) function if*

$$\Phi(t) = \int_0^t \phi(s) ds, \quad t \geq 0,$$

where the real-valued function  $\phi$  defined on  $[0, \infty)$  has the following properties

$$\begin{aligned} \phi(0) &= 0, \quad \phi(s) > 0, \quad s > 0, \quad \lim_{s \rightarrow \infty} \phi(s) = \infty, \\ \phi &\text{ is right continuous and nondecreasing on } [0, \infty). \end{aligned}$$

We define

$$\psi(t) = \sup_{\{\phi(s) \leq t\}} s, \quad t \geq 0, \quad \Psi(t) = \int_0^t \psi(s) ds.$$

Then  $\Psi$  is a Young's function as well. We call  $\Psi$  the complementary Young's function to  $\Phi$ . If  $\Phi$  is complementary to  $\Psi$ , then  $\Psi$  is complementary to  $\Phi$ .

**Definition 2.2** (Orlicz spaces). *Let  $\Omega$  be a domain in  $\mathbb{R}^n$  and let  $\Phi$  be a Young function. The Orlicz class  $K_\Phi(\Omega)$  is the set of all (equivalent classes modulo equality a.e. in  $\Omega$  of) measure functions  $u$  defined on  $\Omega$  that satisfy  $\int_\Omega \Phi(|u(\mathbf{x})|) d\mathbf{x} < \infty$ . The Orlicz space  $L_\Phi(\Omega)$  is the linear hull of the Orlicz class  $K_\Phi(\Omega)$ , that is, the smallest vector space that contains  $K_\Phi(\Omega)$ . The functional*

$$\|u\|_{\Phi(\Omega)} = \inf \left\{ k > 0 \mid \int_\Omega \Phi \left( \frac{|u(\mathbf{x})|}{k} \right) d\mathbf{x} \leq 1 \right\} < \infty$$

is a norm on  $L_\Phi(\Omega)$ . It is called the Luxembourg norm. Thus,  $L_\Phi(\Omega)$  is a Banach space with respect to the Luxembourg norm.

**Definition 2.3** (Cone condition). *Let  $\mathbf{y}$  be a nonzero vector in  $\mathbb{R}^n$ . Let  $\angle(\mathbf{x}, \mathbf{y})$  be the angle between the position vector  $\mathbf{x}$  and  $\mathbf{y}$ . For given such  $\mathbf{y}$ ,  $h > 0$ , and  $k$  satisfying  $0 < k \leq \pi$ , the set*

$$\Lambda = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = 0 \text{ or } 0 < |\mathbf{x}| \leq h, \angle(\mathbf{x}, \mathbf{y}) \leq k/2\}$$

*is called a finite cone of height  $h$ , axis direction  $\mathbf{y}$  and aperture angle  $k$  with vertex at the origin.*

*$\Omega \subset \mathbb{R}^n$  satisfies the cone condition if there exists a finite cone  $\Lambda$ , such that each  $\mathbf{x} \in \Omega$  is the vertex of a finite cone  $\Lambda_{\mathbf{x}}$  contained in  $\Omega$  and congruent to  $\Lambda$ .*

Now, we define

$$M = M(s) := (1+s)\ln(1+s) - s, \quad N = N(s) := e^s - s - 1, \quad H = H(s) := e^{s^2} - 1.$$

Next, we list some basic facts on the Orlicz spaces  $L_M(\Omega)$ ,  $L_N(\Omega)$  and  $L_H(\Omega)$ .

- (a)  $M$  and  $N$  are the complementary Young's functions (see [1, 8.3]).
- (b) Let  $u(\mathbf{x}) \in L_M(\Omega)$  and  $v(\mathbf{x}) \in L_N(\Omega)$ . By virtue of the generalized Hölder's inequality (see [1, 8.11]), we have  $uv \in L^1(\Omega)$  and

$$\left| \int_{\Omega} uv d\mathbf{x} \right| \leq 2 \|u\|_{M(\Omega)} \|v\|_{N(\Omega)}. \quad (2.5)$$

- (c) Let  $\Omega$  be bounded and satisfy the cone condition in  $\mathbb{R}^2$ . By virtue of [1, Theorem 8.12, 8.25 and 8.27], we have for any  $p \geq 1$  that

$$W^{1,2}(\Omega) \hookrightarrow L_H(\Omega) \hookrightarrow L_N(\Omega) \hookrightarrow L^p(\Omega) \quad \text{and} \quad W^{1,2}(\Omega) \hookrightarrow L_N(\Omega). \quad (2.6)$$

- (d) Denote by  $E_N(\Omega)$  the closure of the set of all bounded measurable functions on  $\Omega$  with respect to the Luxembourg norm  $\|\cdot\|_{N(\Omega)}$ . Then, the Orlicz space  $L_M(\Omega)$  is  $E_N$ -weakly compact, i.e., for any sequence  $\{v_n\} \in L_M(\Omega)$  uniformly bounded, there is a subsequence of  $\{v_n\}$ , still denoted by  $\{v_n\}$ , and a  $v \in L_M(\Omega)$ , such that

$$\int_{\Omega} v_n \varphi d\mathbf{x} \rightarrow \int_{\Omega} v \varphi d\mathbf{x} \quad \text{for any } \varphi \in E_N(\Omega)$$

(see [13, 3. Appendix]).

### 2.3. Global Estimates on the Temperature and Velocity

Now, we are in a position derive the global estimates on temperature and velocity. First, we define

$$\tilde{\varrho}^\varepsilon(\cdot, |\mathbf{x}|) = \begin{cases} \varrho^\varepsilon(\cdot, |\mathbf{x}|), & \varepsilon < |\mathbf{x}| < R, \\ 0, & 0 \leq |\mathbf{x}| \leq \varepsilon, \end{cases} \quad \tilde{\theta}^\varepsilon(\cdot, |\mathbf{x}|) = \begin{cases} \theta^\varepsilon(\cdot, |\mathbf{x}|), & \varepsilon < |\mathbf{x}| < R, \\ \theta^\varepsilon(\cdot, \varepsilon), & 0 \leq |\mathbf{x}| \leq \varepsilon, \end{cases}$$

and make use of (2.2)–(2.4) and the definition of Luxemburg norm  $\|\cdot\|_{L_M(\Omega)}$  to deduce that there exists a constant  $C_1(T)$ , such that

$$\|\tilde{\varrho}^\varepsilon(t, |\mathbf{x}|)\|_{L^1(\Omega)} = M_0, \quad \|\tilde{\varrho}^\varepsilon(t, |\mathbf{x}|)\|_{L_M(\Omega)} \leq C_1(T), \quad \forall t \in (0, T), \quad (2.7)$$

$$\int_{\Omega} \tilde{\varrho}^\varepsilon(t, |\mathbf{x}|) \ln \left( 1 + \tilde{\theta}^\varepsilon(t, |\mathbf{x}|) \right) d\mathbf{x} \leq C_1(T), \quad \forall t \in (0, T), \quad (2.8)$$

and

$$\int_0^T \int_{\Omega} \left| \nabla \ln \left( 1 + \tilde{\theta}^\varepsilon(t, \mathbf{x}) \right) \right|^2 d\mathbf{x} dt \leq C_1(T). \quad (2.9)$$

Notice that  $\theta^\varepsilon(t, r)$  is a smooth function in  $(\varepsilon, R)$ , we can easily verify that

$$\ln(1 + \tilde{\theta}^\varepsilon(t, |\mathbf{x}|)) \in L^2(0, T; W^{1,2}(\Omega)). \quad (2.10)$$

Furthermore, by (2.7) and  $L_M \hookrightarrow L^1(\Omega)$ , we infer that there exists a constant  $C_2(\Omega)$ , such that

$$C_2(\Omega) \leq \|\tilde{\varrho}^\varepsilon(t, |\mathbf{x}|)\|_{L_M(\Omega)} \quad \text{for any } t \in (0, T). \quad (2.11)$$

At this stage we shall need two auxiliary results: 1) The first one is a revised version generalized Korn-Poincaré inequality (see [5, Theorem 10.17]) in the case of the Orlicz spaces. The idea of the proof is essentially the same as that used in the proof of [3, Lemma 3.2] under trivial modification. 2) The other one is the revised version Sobolev embedding in two dimensions which will be used to derive bounds of the temperature. This is an idea due to Lions who ever used similar embedding to derive the global integrability of the temperature in the proof of the existence of weak solutions to the stationary problems for the full compressible Navier-Stokes equations in a bounded domain  $\Omega \subset \mathbb{R}^2$  [16, (6.204) in Section 6.11]. These two auxiliary results are formulated in the following two lemmas.

**Lemma 2.2** (Generalized Korn-Poincaré inequality). *Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain satisfying the cone condition. Assume that  $v \in W^{1,2}(\Omega)$ , and  $\varrho \geq 0$  satisfies*

$$0 < C_1 \leq \|\varrho\|_{L(\Omega)}, \quad \|\varrho\|_{L_M(\Omega)} \leq C_2. \quad (2.12)$$

*Then there is a constant  $C_3$  depending solely on  $C_1$  and  $C_2$ , such that*

$$\|v\|_{L^2(\Omega)} \leq C_3(C_1, C_2) \left[ \|\nabla v\|_{L^2(\Omega)} + \int_{\Omega} \varrho |v| d\mathbf{x} \right].$$

**PROOF.** We prove the lemma by contradiction. Suppose that the conclusion of Lemma 2.2 be false, then there would be a sequence  $\{\varrho_n\}_{n=1}^\infty$  of non-negative functions satisfying (2.12) and a sequence  $\{v_n\}_{n=1}^\infty \subset W^{1,2}(\Omega)$ , such that

$$\|v_n\|_{L^2(\Omega)} \geq C_n \left( \|\nabla v_n\|_{L^2(\Omega)} + \int_{\Omega} \varrho_n |v_n| d\mathbf{x} \right), \quad C_n \rightarrow +\infty. \quad (2.13)$$

Setting  $w_n = v_n \|v_n\|_{L^2(\Omega)}^{-1}$ , making use of (2.6) and (2.13), we find that

$$w_n \rightarrow w = \frac{1}{\sqrt{\Omega}} \quad \text{strongly in } L_N(\Omega). \quad (2.14)$$

In view of the hypothesis (2.12), we see that

$$\int_{\Omega} \varrho_n \varphi d\mathbf{x} \rightarrow \int_{\Omega} \varrho \varphi d\mathbf{x} \quad \text{for any } \varphi \in E_N(\Omega). \quad (2.15)$$

Thus, by virtue of (2.14), (2.15) and (2.5), one has

$$\lim_{n \rightarrow \infty} \int_{\Omega} (\varrho_n w_n - \varrho w) d\mathbf{x} = \lim_{n \rightarrow \infty} \int_{\Omega} \varrho_n (w_n - w) d\mathbf{x} + \lim_{n \rightarrow \infty} \int_{\Omega} (\varrho_n - \varrho) w d\mathbf{x} = 0. \quad (2.16)$$

The identity (2.16), together with (2.12) and (2.14), yields

$$\lim_{n \rightarrow \infty} \int_{\Omega} \varrho_n w_n d\mathbf{x} = \int_{\Omega} \varrho w d\mathbf{x} > 0. \quad (2.17)$$

On the other hand, (2.13) implies

$$\lim_{n \rightarrow \infty} \int_{\Omega} \varrho_n w_n d\mathbf{x} = 0, \quad (2.18)$$

which contradicts with (2.17). Therefore, the conclusion of Lemma 2.2 remains true.  $\square$



**Lemma 2.3** (Sobolev embedding). *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$  and*

$$\Theta = \{\theta \mid \|\ln(1 + \theta)\|_{W^{1,2}(\Omega)} \leq C_1\}. \quad (2.19)$$

*Then, for any  $q \geq 1$ , there is a constant  $C_2$  depending solely on  $q$ ,  $C_1$  and  $\Omega$ , such that*

$$\|\theta\|_{L^q(\Omega)} \leq C_2(q, C_1, \Omega) \quad \text{for all } \theta \in \Theta.$$

PROOF. We use (2.19) and (2.6) to infer that

$$\Lambda := \|\ln(1 + \theta)\|_{L_H(\Omega)} \leq C_3(\Omega) \|\ln(1 + \theta)\|_{W^{1,2}(\Omega)} \leq C_3 C_1.$$

Easily, it suffices to consider the case  $\Lambda \neq 0$ . By the definition of the Luxemburg norm  $\|\cdot\|_{L_N(\Omega)}$ , we obtain

$$\int_{\Omega} \left\{ \exp[\Lambda^{-1} \ln(1 + \theta)]^2 - 1 \right\} d\mathbf{x} \leq 1,$$

which yields

$$\int_{\Omega} (1 + \theta)^{\Lambda^{-2} \ln(1 + \theta)} d\mathbf{x} \leq 1 + |\Omega|.$$

Hence,

$$\begin{aligned} \int_{\Omega} \theta^q d\mathbf{x} &\leq \int_{\{\theta \leq e^{q\Lambda^2} - 1\}} \theta^q d\mathbf{x} + \int_{\{\theta > e^{q\Lambda^2} - 1\}} (1 + \theta)^q d\mathbf{x} \\ &\leq (e^{q\Lambda^2} - 1)^q |\Omega| + \int_{\{\theta > e^{q\Lambda^2} - 1\}} (1 + \theta)^{\Lambda^{-2} \ln(1 + \theta)} d\mathbf{x} \\ &\leq \left[ e^{q(C_1 C_3)^2} - 1 \right]^q |\Omega| + 1 + |\Omega| := C_2(q, C_1, \Omega). \end{aligned}$$

This completes the proof of Lemma 2.3.  $\square$

Thus, with the help of Lemma 2.3 and the estimates (2.7)–(2.11), we conclude

$$\|\ln(1 + \tilde{\theta}^\varepsilon(t, |\mathbf{x}|))\|_{L^2((0,T), W^{1,2}(\Omega))} \leq C_3(T, M_0, \Omega),$$

which together with Lemma 2.3 gives

$$\|\theta^\varepsilon(t, |\mathbf{x}|)\|_{L^2((0,T), L^q(\Omega^\varepsilon))} \leq \|\tilde{\theta}^\varepsilon(t, |\mathbf{x}|)\|_{L^2((0,T), L^q(\Omega))} \leq C_4(T, M_0, q, \Omega) \quad (2.20)$$

for any  $q \in [1, \infty)$ , where  $\Omega^\varepsilon = \{\mathbf{x} \in \mathbb{R}^2 \mid \varepsilon < |\mathbf{x}| < R\}$ . Furthermore, using (2.4), (2.20) and Hölder's inequality, we get

$$\begin{aligned} \int_0^T \int_{\Omega^\varepsilon} |\nabla \theta^\varepsilon| d\mathbf{x} dt &\leq \left( \int_0^T \int_{\Omega^\varepsilon} \frac{|\nabla \theta^\varepsilon|^2}{(\theta^\varepsilon)^2(t, |\mathbf{x}|)} d\mathbf{x} dt \right)^{1/2} \left( \int_0^T \int_{\Omega^\varepsilon} (\theta^\varepsilon)^2(t, |\mathbf{x}|) d\mathbf{x} dt \right)^{1/2} \\ &\leq \left( \int_0^T \int_\varepsilon^R \frac{(\theta_r^\varepsilon)^2}{(\theta^\varepsilon)^2(t, r)} r dr dt \right)^{1/2} C_4^{1/2}(T, M_0, 2, \Omega) \\ &\leq C_5(T, M_0, \Omega). \end{aligned} \quad (2.21)$$

Now, we denote  $\mathbf{u}^\varepsilon(t, \mathbf{x}) := u^\varepsilon(t, |\mathbf{x}|)\mathbf{x}/|\mathbf{x}|$ , which is a smooth function in  $\Omega^\varepsilon$ . By a simple calculation, we find that

$$|\nabla \mathbf{u}^\varepsilon|^2 = (u_r^\varepsilon)^2 + \frac{(u^\varepsilon)^2}{r^2}, \quad (u_r^\varepsilon)^\zeta + \frac{u^\zeta}{r^\zeta} \leq 2^{\zeta/2} |\nabla \mathbf{u}^\varepsilon|^\zeta \quad \text{for any } \zeta \geq 1.$$

If we utilize (2.4), (2.20) with  $q = 2$  and Hölder's inequality, we obtain

$$\begin{aligned}
\int_0^T \int_{\Omega^\varepsilon} |\nabla \mathbf{u}^\varepsilon|^{4/3} d\mathbf{x} dt &\leq \left( \int_0^T \int_{\Omega^\varepsilon} \frac{|\nabla \mathbf{u}^\varepsilon|^2}{\theta^\varepsilon(t, |\mathbf{x}|)} d\mathbf{x} dt \right)^{2/3} \left( \int_0^T \int_{\Omega^\varepsilon} (\theta^\varepsilon)^2(t, |\mathbf{x}|) d\mathbf{x} dt \right)^{1/3} \\
&\leq \left( \frac{1}{2\mu} \right)^{2/3} \left( \int_0^T \int_\varepsilon^R \frac{\mathcal{Q}^\varepsilon}{\theta^\varepsilon(t, r)} r dr dt \right)^{2/3} \left[ \int_0^T \int_\varepsilon^R (\theta^\varepsilon)^2(t, r) r dr dt \right]^{1/3} \\
&\leq C_6(T, M_0, \Omega).
\end{aligned} \tag{2.22}$$

Since  $\mathbf{u}^\varepsilon(\cdot, \mathbf{x})|_{\partial\Omega^\varepsilon} = 0$ , we extend  $\mathbf{u}^\varepsilon$  by zero outside  $\Omega^\varepsilon$ , and employ Sobolev's inequality and (2.22) to deduce that

$$\int_0^T \left( \int_{\Omega^\varepsilon} |\mathbf{u}^\varepsilon|^4 d\mathbf{x} \right)^{1/3} dt \leq C_7(T, M_0, \Omega). \tag{2.23}$$

Finally, combining (2.20)-(2.22) with (2.23), we conclude

**Lemma 2.4** (Global estimates of  $u$  and  $\theta$ ). *Under the assumption of Lemma 2.1, there are constants  $C_1$  and  $C_2$ , such that*

$$\int_0^T \left( \int_\varepsilon^R (\theta^\varepsilon)^q(t, r) r dr \right)^{2/q} dt \leq C_1(T, M_0, q, \Omega), \tag{2.24}$$

$$\int_0^T \left( \int_\varepsilon^R |\theta_r^\varepsilon(t, r)| r dr \right) dt \leq C_2(T, M_0, \Omega), \tag{2.25}$$

$$\int_0^T \int_\varepsilon^R \left( |u_r^\varepsilon|^{4/3} + \left| \frac{u^\varepsilon}{r} \right|^{4/3} \right) r dr dt \leq C_2(T, M_0, \Omega), \tag{2.26}$$

$$\int_0^T \left( \int_\varepsilon^R (u^\varepsilon)^4 r dr \right)^{1/3} dt \leq C_2(T, M_0, \Omega). \tag{2.27}$$

### 3. Local Estimates

In order to taking to the limit as  $\varepsilon \rightarrow 0$ , we will need further uniform bounds of higher order derivatives. Such bounds will be obtained away from the origin of Lagrangian space in the following sense. Define a curve  $r_h^\varepsilon(t)$  for  $h \geq 0$  by

$$h = \int_\varepsilon^{r_h^\varepsilon(t)} \varrho^\varepsilon(t, r) r dr. \tag{3.1}$$

Easily, by (1.7),

$$\frac{\partial r_h^\varepsilon}{\partial t} = u^\varepsilon(t, r_h^\varepsilon).$$

Thus  $r_h^\varepsilon(t)$  is the position at time  $t$  of a fixed fluid particle. Furthermore, an easy estimate, based on Jensen's inequality and boundedness of  $\int_\varepsilon^R \varrho^\varepsilon \Psi(\varrho_\varepsilon^{-1}) r dr$  in (2.4) (see (1.16)), shows that  $h \rightarrow 0$  at a uniform rate as  $r_h^\varepsilon(t) \rightarrow 0$ . That is, given  $h > 0$ , there is a positive constant  $C = C(h)$  independently of  $\varepsilon$  and  $T$ , such that

$$r_h^\varepsilon(t) \geq C(h)^{-1}. \tag{3.2}$$

Using (3.2), we can derive pointwise bounds for the approximate density and temperature, which are valid away from the origin  $h = 0$  of Lagrangian space, but independent of  $\varepsilon$ . The idea of deriving the pointwise boundedness was used first by Kazhikov and Shelukhin [14], and later adapted by Frid and Shelukhin [7], and by Hoff and Jenssen [9] in a nontrivial way to show a pointwise boundedness similar to that given in Lemma 3.1 below.

**Lemma 3.1** (Pointwise bounds). *Given  $h > 0$  and  $T > 0$ , there is a constant  $C = C(T, h)$ , independent of  $\varepsilon$ , such that, if  $r_h^\varepsilon(t)$  is given by (3.1), then*

$$C^{-1} \leq \varrho^\varepsilon(t, r) \leq C \quad \text{for } r \in [r_h^\varepsilon(t), R] \text{ and } t \in [0, T],$$

and

$$\int_0^t \|\theta^\varepsilon(\tau, \cdot)\|_{h, \infty} d\tau \leq C$$

where  $\|\cdot\|_{h, \infty}$  denotes the  $L^\infty$ -norm over  $[r_h^\varepsilon(t), R]$ .

PROOF. Taking  $n = 2$  and  $m = 1$  in the proof of [9, Lemma 2], we immediately obtain Lemma 3.1.  $\square$

Next, we shall make use of a cut-off function which is convected with the flow and vanishes near the origin. The cut-off function is constructed as follows: For given  $\varepsilon$  and  $h$ , we can fix a smooth, increasing function  $\phi_0(r)$  with  $\phi_0(r) \equiv 0$  on  $[0, r_h(0)]$ ,  $0 < \phi_0(r) \leq 1$  on  $(r_h(0), 2r_h(0))$  and  $\phi_0(r) \equiv 1$  on  $[2r_h(0), R]$ , and then define  $\phi(t, r)$  to be the solution of the problem

$$\phi_t + u\phi_r = 0, \quad \phi(0, r) = \phi_0(r). \quad (3.3)$$

We choose  $\phi_0$  so that

$$\phi_0'(r) \leq C(h)\phi_0^{(p-1)/p}(r) \quad \text{for some } p > 2.$$

Thus, we can easily show that this boundedness persists for all time, i.e.,

$$|\phi_r(t, r(t))| \leq C(T, h)\phi(t, r(t))^{(p-1)/p}. \quad (3.4)$$

We shall take  $p$  so large that the exponent on the right-hand side of (3.4) is close to one. Notice that here we have suppressed the dependence of  $\phi$  on  $\varepsilon$  and  $h$ .

As in [9], we now introduce three functionals of higher-order derivatives for  $(u_r^\varepsilon, \theta_r^\varepsilon)$ :

$$\begin{aligned} \mathcal{A}(T) &:= \sup_{0 \leq t \leq T} \sigma(t) \int_{r_h(t)}^R \phi(t, r) \left( u_r^\varepsilon + \frac{u^\varepsilon}{r} \right)^2(t, r) r dr \\ &\quad + \int_0^T \int_{r_h(t)}^R \sigma(t) \phi(t, r) (\dot{u}^\varepsilon)^2(t, r) r dr dt, \\ \mathcal{B}(T) &:= \sup_{0 \leq t \leq T} \sigma(t) \int_{r_h(t)}^R \phi(t, r) (\theta_r^\varepsilon)^2(t, r) r dr + \int_0^T \int_{r_h(t)}^R \sigma(t) \phi(t, r) (\dot{\theta}_r^\varepsilon)^2(t, r) r dr dt, \\ \mathcal{D}(T) &:= \sup_{0 \leq t \leq T} \sigma^2(t) \int_{r_h(t)}^R \phi^2(t, r) (\theta_r^\varepsilon)^2(t, r) r dr + \int_0^T \int_{r_h(t)}^R \sigma^2(t) \phi^2(t, r) (\dot{\theta}_r^\varepsilon)^2(t, r) r dr dt, \end{aligned}$$

where  $\sigma(t) = \min\{1, t\}$ , “dot” denotes the convective derivative  $\partial_t + u\partial_r$ , and we have again suppressed the dependence on  $\varepsilon$  and  $h$  for simplicity. Thus, we have the following estimates.

**Lemma 3.2** (Higher order boundedness). *Let  $h > 0$  and  $T > 0$  be given. Then there is a constant  $C = C(T, h)$ , such that*

$$\int_0^T \int_{r_h(t)}^R \phi \left( u_r^\varepsilon + \frac{u^\varepsilon}{r} \right)^2(t, r) r dr dt \leq C(T, h).$$

and

$$\mathcal{A}(T), \mathcal{B}(T), \mathcal{D}(T) \leq C(T, h).$$

PROOF. This lemma can be shown following the same procedure as in the proof of [9, Lemmas 4 and 6] with taking  $m = 1$ ,  $v = 0$  and  $w = 0$ . We should point out here that in the proof one should make use of Lemma 2.1, Lemma 3.1, (3.3) and (3.4).  $\square$

As the end of this section, we give some uniform integrability estimates. To describe these, we define the strictly increasing, convex function  $G$  by

$$G : [1, \infty) \rightarrow [0, \infty), \quad G(y) := y \log y.$$

Then  $G^{-1} : [0, \infty) \rightarrow [1, \infty)$ , and one can define for  $r, c > 0$  the function

$$\omega(r, c) := r + rG^{-1}\left(\frac{c}{r}\right). \quad (3.5)$$

It is easy to see that for each fixed  $c$  the function  $r \mapsto \omega(r, c)$  is continuous and increasing on  $(0, \infty)$ , and that

$$\lim_{r \rightarrow 0} \omega(r, c) = 0.$$

Finally, if  $E \subset [0, R]$ , we define  $|E| := \int_E r dr$ .

**Lemma 3.3** (Uniform integrability). *Let  $\omega$  be the same as in (3.5).*

(a) *Given  $b > 0$  and  $T > 0$ , there is a constant  $C = C(T, b)$ , such that*

$$\int_{t_1}^{t_2} \left( \left\| \frac{u}{\theta^{1/2}} \right\|_{b, \infty} + \|\log(\theta \vee 1)\|_{b, \infty} \right) dt \leq C(T, b) \quad \text{for any } t_1, t_2 \in [0, T], \quad (3.6)$$

where  $\theta \vee 1 = \max\{\theta, 1\}$ .

(b) *If  $\varepsilon \geq 0$ , and  $\varrho : [\varepsilon, R] \rightarrow \mathbb{R}$  is strictly positive and satisfies*

$$\left| \int_{\varepsilon}^R \varrho \log \varrho \, r dr \right| \leq C.$$

*Then, for any measure set  $E \subset [\varepsilon, R]$ ,*

$$\left| \int_E \varrho \, r dr \right| \leq \omega(|E|, C).$$

(c) *Let  $b > 0$  and  $T > 0$ . Then there is a constant  $C = C(T, b)$  such that, if for  $t \in [0, T]$ ,  $E(t)$  is a measurable subset of  $[b, R]$ , and if  $(\varrho, u, \theta) = (\varrho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$  is the approximate solution described at the beginning of Section 2, then*

$$\begin{aligned} \int_0^T \int_{E(t)} \varrho \theta r dr dt &\leq \omega \left( \int_0^T \int_{E(t)} \varrho r dr dt, C(T, b) \right), \\ \int_0^T \int_{E(t)} \varrho u^2 r dr dt &\leq C(T, b) \omega \left( \int_0^T \int_{E(t)} \varrho r dr dt, C(T, b) \right)^{1/4}. \end{aligned}$$

PROOF. The estimate (3.6) is a consequence of the entropy estimate (2.4). The uniform integrability bounds in (b) and (c), which are important in showing the limit of these approximate solutions to be a weak solution in Section 4, can be obtained using arguments similar to those used in [9, Lemmas 8 and 9], and hence we omit the proof here.  $\square$

#### 4. Proof of Theorem 1.1

By virtue of the a priori estimates derived in Sections 2 and 3, we are now able to prove our main theorem by taking appropriate limits in a manner analogous to that in [9].

To begin with, we denote by  $H_\delta$  a standard mollifier (in  $r$ ) of width  $\delta$ , and for  $\varepsilon > \delta$  we define the smooth approximate initial data  $(\varrho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, \theta_0^{\varepsilon,\delta})$  to  $(\varrho_0, u_0, \theta_0)$  as follows:

1) Extend  $\varrho_0$  by its average value outside  $[\varepsilon, R]$ , mollify with  $H_\delta$ , restrict to  $[\varepsilon, R]$ , and then multiply by a constant to normalize the total mass to be  $M_0 = \int_0^R \varrho_0 r dr$ . The resulting density function is denoted by  $\varrho_0^{\varepsilon,\delta}(r)$ .

2) Redefine  $u_0$  to be zero on  $[0, 2\varepsilon]$  and  $[R - 2\delta, R]$ , then mollify with  $H_\delta$  to get  $u_0^{\varepsilon,\delta}$ . Note that  $u_0^{\varepsilon,\delta}$  is identically zero in a neighborhood of  $r = \varepsilon$  or  $R$ .

3) Redefine  $\theta_0$  to be its average value on  $[0, 2\varepsilon]$  and  $[R - 2\delta]$ , then mollify with  $H_\delta$  to get  $\theta_0^{\varepsilon,\delta}$ . Note that  $\theta_0^{\varepsilon,\delta}$  is constant in a neighborhood of  $r = \varepsilon$  or  $R$ .

The resulting data  $\varrho_0^{\varepsilon,\delta}, u_0^{\varepsilon,\delta}, \theta_0^{\varepsilon,\delta}$  then satisfy the hypotheses (1.14) and (1.15) with the constants independent of  $\varepsilon$  and  $\delta$ . Thus, there is a global-in-time smooth solution  $(\varrho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})$  of the system (1.7)–(1.9) with the initial boundary conditions (1.12) and (2.1). This is a result of Frid and Shelukhin's work [7] in the annular domain  $(\varepsilon, R)$ . Next, we want to pass to the limit to get a global weak solution. As in (3.1), we define the particle path  $r_h^{\varepsilon,\delta}(t)$  associated with the approximate solution  $(\varrho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})$  by

$$h = \int_\varepsilon^{r_h^{\varepsilon,\delta}(t)} \varrho^{\varepsilon,\delta}(t, r) r dr, \quad h, \varepsilon, \delta > 0. \quad (4.1)$$

##### 4.1. Convergence of the Approximate Solutions

By the a priori estimates established in Lemmas 3.1–3.2, 2.1 and 2.4, we have the following three propositions, which imply that there is a subsequence  $(\varepsilon_j, \delta_j) \rightarrow (0, 0)$ , such that the approximate solutions and their associated particle paths are convergent.

**Proposition 4.1.** *Let  $(\varrho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \theta^{\varepsilon,\delta})$  and  $r_h^{\varepsilon,\delta}(t)$  be as described above.*

- (a) *There is a subsequence  $(\varepsilon_j, \delta_j) \rightarrow (0, 0)$ , such that  $r_h^{\varepsilon,\delta}(t)$  converges uniformly for  $(t, h)$  in any compact subset of  $[0, \infty) \times (0, M_0)$ , and the limit  $r_h(t)$  is Hölder-continuous in  $(t, h)$  on any compact set.*
- (b) *If  $\underline{r}(t) := \lim_{h \rightarrow 0} r_h(t)$ , then  $\underline{r}(t) \in C_{\text{loc}}^{0,1/4}[0, \infty)$  and  $\lim_{t \rightarrow 0} \underline{r}(t) = 0$ .*
- (c) *If the “fluid region”  $\mathcal{F}$  is defined by*

$$\mathcal{F} := \{(t, r) \mid \underline{r}(t) < r \leq R, \ 0 \leq t < \infty\},$$

*then  $\mathcal{F} \cap \{t > 0\} \cap \{r < R\}$  is an open set.*

**Proposition 4.2.** *Let the hypotheses of Proposition 4.1 be satisfied. Then there is a further subsequence, still denoted by  $(\varepsilon_j, \delta_j)$ , and limiting functions  $u$  and  $\theta$ , such that*

$$u^{\varepsilon_j, \delta_j} \rightarrow u, \quad \theta^{\varepsilon_j, \delta_j} \rightarrow \theta$$

*uniformly on any compact subset of  $\mathcal{F} \cap \{t > 0\}$ . The functions  $u$  and  $\theta$  are Hölder-continuous on any compact set. Furthermore, for any  $T > 0$ ,*

$$u^{\varepsilon_j, \delta_j} \rightarrow u \text{ weakly in } L^{4/3}((0, T), W_{\text{loc}}^{1,4/3}(0, R]) \quad (4.2)$$

**Proposition 4.3.** *Assume that the hypotheses of Proposition 4.2 hold. Then there is a further subsequence  $(\varepsilon_j, \delta_j) \rightarrow (0, 0)$  and a function  $\varrho(t, r)$  such that*

$$\varrho^{\varepsilon_j, \delta_j}(t, \cdot) \rightarrow \varrho(t, \cdot) \text{ in } H^{-1}([\underline{r}(t) + \eta, R], r dr)$$

and

$$\varrho^{\varepsilon_j, \delta_j}(t, \cdot) \rightharpoonup \varrho(t, \cdot) \text{ in } L^2([\underline{r}(t) + \eta, R], r dr)$$

for all  $t \in [0, T]$  and all  $\eta > 0$ . In addition, if  $\varrho^{\varepsilon_j, \delta_j}(t, \cdot)$  is taken to be zero for  $r \leq \varepsilon_j$ , then

$$\varrho^{\varepsilon_j, \delta_j}(t, \cdot) \rightarrow 0 \text{ in } L^1([0, \underline{r}(t)], r dr) \quad \text{when } \underline{r}(t) > 0.$$

Also, for  $h > 0$  and  $T > 0$ , there is constant  $C = C(T, h)$ , such that

$$C^{-1}(T, h) \leq \varrho \leq C(T, h) \quad \text{for } 0 \leq t \leq T \text{ and } r_h(t) \leq r \leq R.$$

Finally, for  $h > 0$  and  $t \geq 0$ ,

$$h = \int_{\underline{r}(t)}^{r_h(t)} \varrho(t, r) r dr. \quad (4.3)$$

By virtue of the a priori estimates established in Sections 2 and 3, one can show Propositions 4.1–4.3 in the same manner as that in the proof of [9, Propositions 1–3], except (4.2) of Proposition 4.2 which are obtained by applying the uniform global estimates given in Lemma 2.4. In addition, the identity (4.3) shows that mass is conserved for the limiting solution

$$M_0 = \int_{\underline{r}(t)}^R \varrho(t, r) r dr = \int_0^R \varrho_0(r) r dr. \quad (4.4)$$

#### 4.2. Weak Forms of the Navier-Stokes Equations

We now turn to the proof that the limiting functions are indeed a weak solution of the Navier-Stokes equations in  $[0, \infty) \times \Omega$  in the sense of Theorem 1.1.

First, the limiting functions  $\varrho$ ,  $u$  and  $\theta$  have been defined in the fluid region  $\mathcal{F}$  but not elsewhere. We therefore define  $\varrho$ ,  $\varrho u$  and  $\varrho \theta$  to be identically zero in the vacuum region  $\mathcal{F}^c$ . As in Section 1 we let  $r(\mathbf{x}) = |\mathbf{x}|$  and define the velocity vector  $\mathbf{u} : [0, \infty) \times \bar{\Omega}$  by

$$\mathbf{u}(t, \mathbf{x}) = u(t, r) \frac{\mathbf{x}}{r}. \quad (4.5)$$

Abusing notation slightly, we also write  $\varrho(t, \mathbf{x})$  and  $\theta(t, \mathbf{x})$  in place of  $\varrho(t, r(\mathbf{x}))$  and  $\theta(t, r(\mathbf{x}))$ . Similar notation applies to the approximate solutions, for which we now write  $u^j$  in place of  $u^{\varepsilon_j, \delta_j}$ , etc.

We first show that  $(\varrho, \mathbf{u}, \theta)$  satisfies the weak form (1.17) of the mass equation.

**Proposition 4.4.** *Let  $(\varrho, \mathbf{u}, \theta)$  be the limit described above in Propositions 4.1–4.3. Then,*

- (a) *The weak form (1.17) of the mass equation holds for any  $C^1$  test function  $\phi : [t_1, t_2] \times \bar{\Omega} \rightarrow \mathbb{R}$ .*
- (b)  *$\varrho \in C([0, \infty), W^{1, \infty}(\Omega))^*$ .*
- (c)  *$\varrho^{1/2} \mathbf{u} \in L^\infty([0, \infty), L^2(\Omega))$ .*
- (d)  *$\mathbf{u} \in L^{4/3}(0, T; W^{1, 4/3}(\Omega))$ .*

PROOF. The assertions (a)–(c) follow from the analogous arguments as in the proof of [9, Proposition 4], where we have made use of Lemma 3.3 and Proposition 4.1–4.3.

To show (d), we use (2.26), (2.27), (4.2) and the lower semi-continuity to deduce that

$$\int_0^T \int_0^R \left( |u_r|^{4/3} + \left| \frac{u}{r} \right|^{4/3} \right) r dr dt \leq \lim_{b \rightarrow 0} \lim_{j \rightarrow \infty} \int_0^T \int_b^R \left( |u_r^j|^{4/3} + \left| \frac{u^j}{r} \right|^{4/3} \right) r dr dt \leq C(T) \quad (4.6)$$

and

$$\int_0^T \left( \int_0^R u^4 r dr \right)^{1/3} dt \leq \lim_{b \rightarrow 0} \lim_{j \rightarrow \infty} \int_0^T \left( \int_b^R u_j^4 r dr \right)^{1/3} dt \leq C(T).$$

We can compute that

$$\partial_{x_j} \left[ u(t, |\mathbf{x}|) \frac{x_i}{|\mathbf{x}|} \right] = u_r(t, |\mathbf{x}|) \frac{x_i x_j}{|\mathbf{x}|^2} + u(t, |\mathbf{x}|) \left( \frac{\delta_{ij}}{|\mathbf{x}|} - \frac{x_i x_j}{|\mathbf{x}|^3} \right), \quad 1 \leq i, j \leq 2, \quad (4.7)$$

thus,  $|\nabla \mathbf{u}|^{4/3} \leq 2^{2/3} (|u_r|^{4/3} + |u/r|^{4/3})$  and  $|\mathbf{u}|^2 = u^2$ . Hence, we find that  $\mathbf{u} \in L^{4/3}(0, T; W^{1,4/3}(\Omega))$ . This completes the proof.  $\square$

Next, we show that the weak form of the momentum equations in the spherically symmetric case holds.

**Lemma 4.1.** *Let  $\varrho, u$  and  $\theta$  be the functions given in Propositions 4.2 and 4.3. Let  $t_1 < t_2$  and  $\phi$  be a  $C^1$ -function on  $[t_1, t_2] \times [0, R]$ , such that  $\phi(t, 0) = \phi(t, R) = 0$  for  $t \in [t_1, t_2]$ . Then, the following identity holds.*

$$\begin{aligned} & \int_0^R \varrho u \phi r dr \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^R \left[ \varrho u \phi_t + \varrho u^2 \phi_r + P(\varrho, \theta) \left( \phi_r + \frac{\phi}{r} \right) \right] r dr dt \\ &= \int_{t_1}^{t_2} \int_0^R \varrho f \phi r dr dt - \nu \int_{t_1}^{t_2} \int_0^R \left( u_r + \frac{u}{r} \right) \left( \phi_r + \frac{\phi}{r} \right) r dr dt. \end{aligned} \quad (4.8)$$

PROOF. We first consider a simpler case in which the test function vanishes in a neighborhood of the origin. Assume that  $\psi$  is a  $C^1$ -function on  $[t_1, t_2] \times [0, R]$  satisfying  $\psi \equiv 0$  on  $[0, b]$  for some  $b > 0$ . Then, applying the proof of [9, Lemma 10 (a)] and combining with the weak convergence (4.2), we can easily show that the weak form of the momentum equation (1.8) holds for the test function  $\psi$ :

$$\begin{aligned} & \int_0^R \varrho u \psi r dr \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^R \left[ \varrho u \psi_t + \varrho u^2 \psi_r + P \left( \psi_r + \frac{\psi}{r} \right) \right] r dr dt \\ &= \int_{t_1}^{t_2} \int_0^R \varrho f \psi r dr dt - \nu \int_{t_1}^{t_2} \int_0^R \left( u_r + \frac{u}{r} \right) \left( \psi_r + \frac{\psi}{r} \right) r dr dt. \end{aligned} \quad (4.9)$$

To extend the identity (4.9) to the case that test functions are supported in  $[0, R]$ , we fix an increasing  $C^1$  function  $\chi : [0, \infty) \rightarrow [0, 1]$  with  $\chi \equiv 0$  on  $[0, 1]$  and  $\chi \equiv 1$  on  $[2, \infty)$ , and define  $\chi^b(r) := \chi(r/b)$  for  $b > 0$ . Let  $\phi$  be a  $C^1$  function on  $[t_1, t_2] \times [0, R]$  such that  $\phi(t, 0) = \phi(t, R) = 0$  for  $t \in [t_1, t_2]$ , and define  $\phi^b := \chi^b \phi$ . Then, the previous lemma applies to the test functions  $\phi^b = \chi^b \phi$ . We obtain

$$\begin{aligned} & \int_0^R \varrho u \chi^b \phi r dr \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^R \left[ \varrho u \chi^b \phi_t + \varrho u^2 (\chi^b \phi)_r + P(\chi^b \phi)_r + \frac{P \chi^b \phi}{r} \right] r dr dt \\ &= \int_{t_1}^{t_2} \int_0^R \varrho f \chi^b \phi r dr dt - \nu \int_{t_1}^{t_2} \int_0^R \left( u_r + \frac{u}{r} \right) \left[ (\chi^b \phi)_r + \frac{\chi^b \phi}{r} \right] r dr dt. \end{aligned} \quad (4.10)$$

The first, second, fifth and sixth terms in (4.10) converge to the corresponding terms in (4.8) as  $b \rightarrow 0$  by virtue of the dominated convergence theorem. For the third term we have

$$\int_{t_1}^{t_2} \int_0^R \varrho u^2 (\chi^b \phi)_r r dr dt = \int_{t_1}^{t_2} \int_0^R \varrho u^2 (\chi_r^b \phi + \chi^b \phi_r) r dr dt, \quad (4.11)$$

and the second term on the right-hand side of (4.10) clearly tends to the third term in (4.8) as  $b \rightarrow 0$ . Since  $\phi(t, 0) = 0$ , we can write  $\phi(t, r) = r\varphi(t, r)$  for some smooth  $\varphi$ . Thus, due to  $|\partial_r \chi^b| \leq C/b$  we can bound the first term on the right-hand side of (4.11) by

$$\int_{t_1}^{t_2} \int_0^b \varrho u^2 C \frac{r}{b} |\varphi(r)| r dr dt \leq C \int_{t_1}^{t_2} \int_b^{2b} \varrho u^2 r dr dt, \quad (4.12)$$

which goes to zero as  $b \rightarrow 0$  by utilizing the boundedness of the limiting energy. Moreover, the same argument applies to the fourth term in (4.10). Finally, for the last term on the right-hand side of (4.10), we have

$$\int_{t_1}^{t_2} \int_0^R \left(u_r + \frac{u}{r}\right) \left(\phi_r^b + \frac{\phi^b}{r}\right) r dr dt = \int_{t_1}^{t_2} \int_0^R \left(u_r + \frac{u}{r}\right) \left(\phi \chi_r^b + \phi_r \chi^b + \frac{\chi^b \phi}{r}\right) r dr dt. \quad (4.13)$$

Similarly to (4.12), we deduce that

$$\left| \int_{t_1}^{t_2} \int_0^R \left(u_r + \frac{u}{r}\right) \phi \chi_r^b r dr dt \right| \leq C \int_{t_1}^{t_2} \int_b^{2b} \left(u_r + \frac{u}{r}\right) r dr dt, \quad (4.14)$$

which tends to zero as  $b \rightarrow 0$  by (4.6). On the other hand, letting  $b \rightarrow 0$  in (4.13), using the dominated convergence theorem and (4.14), we conclude

$$\int_{t_1}^{t_2} \int_0^R \left(u_r + \frac{u}{r}\right) \left(\phi_r \chi^b + \frac{\phi \chi^b}{r}\right) r dr dt \rightarrow \int_{t_1}^{t_2} \int_0^b \left(u_r + \frac{u}{r}\right) \left(\phi_r + \frac{\phi}{r}\right) r dr dt.$$

This completes the proof of the lemma.

**Remark 4.1.** Applying (2.26)-(2.27) and the proof of Lemma 4.1 to [9, Lemma 11], we can see that

$$\lim_{R \rightarrow 0} \lim_{j \rightarrow \infty} \mathcal{U}(j, \phi^R) = \nu \int_{t_1}^{t_2} \int_0^b \left(u_r + \frac{u}{r}\right) \left(\phi_r + \frac{\phi}{r}\right) r dr dt$$

still holds for cylindrically symmetric case in [9, Lemma 11].

Now, we are able to show that the weak form (1.18) for the momentum equations in Cartesian coordinates is satisfied.

**Proposition 4.5.** *The weak form (1.18) of the momentum equations, as stated in Theorem 1.1 (d), holds.*

PROOF. Given  $\psi$  as described in the theorem, we define

$$\phi(t, r) := \int_S \psi(t, r\mathbf{y}) y_i dS_{\mathbf{y}} \text{ for fixed } i = 1, 2,$$



where  $S \subset \mathbb{R}^2$  denotes the unit circle. Then,  $\phi(t, 0) = \phi(t, R) \equiv 0$ . It thus follows from Lemma 4.1 that

$$\begin{aligned} & \int_0^R \varrho u \phi r dr \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_0^R \left[ \varrho u \phi_t + \varrho u^2 \phi_r + P \left( \phi_r + \frac{\phi}{r} \right) \right] r dr dt \\ &= \int_{t_1}^{t_2} \int_0^R \varrho f \phi r dr dt - \nu \int_{t_1}^{t_2} \int_0^R \left( u_r + \frac{u}{r} \right) \left( \phi_r + \frac{\phi}{r} \right) r dr dt. \end{aligned} \quad (4.15)$$

We convert each of the terms in (4.15) to integrals in Cartesian coordinates involving  $\psi$ . The treatment of the terms involving derivations are very much similar to those in the proof of Proposition 4.4 (see [9, Proposition 4]), except for the last term, which we deal with in details. We may rewrite the last term of (4.15) as

$$\begin{aligned} & \int_{t_1}^{t_2} \int_0^R \left( u_r + \frac{u}{r} \right) \left( \phi_r + \frac{\phi}{r} \right) r dr dt \\ &= \int_{t_1}^{t_2} \int_0^R \left[ \left( \frac{u}{r} \right)_r \phi_r r^2 + \frac{u}{r} (r\phi)_r \right] dr dt \\ &= \int_{t_1}^{t_2} \int_0^R r \left( \frac{u}{r} \right)_r \left( \int_S \psi_{x_k}(t, r\mathbf{y}) y_i y_k dS_{\mathbf{y}} \right) r dr dt \\ &\quad + \int_{t_1}^{t_2} \int_0^R \frac{u}{r} \left( \int_S \psi_{x_i}(t, r\mathbf{y}) dS_{\mathbf{y}} \right) r dr dt \\ &= \int_{t_1}^{t_2} \int_{\Omega} \left[ \left( \frac{u(t, |\mathbf{x}|)}{|\mathbf{x}|} \right)_r \frac{x_k x_i}{|\mathbf{x}|} + \frac{u}{|\mathbf{x}|} \delta_{ik} \right] \psi_{x_k}(t, \mathbf{x}) r d\mathbf{x} dt \\ &= \int_{t_1}^{t_2} \int_{\Omega} \left( u(t, |\mathbf{x}|) \frac{x_i}{|\mathbf{x}|} \right)_{x_k} \psi_{x_k}(t, r\mathbf{y}) r d\mathbf{x} dt \\ &= \int_{t_1}^{t_2} \int_{\Omega} \nabla u_i(t, |\mathbf{x}|) \nabla \psi d\mathbf{x} dt, \end{aligned} \quad (4.16)$$

where the repeated indexes should be summed. Notice that  $\partial_{x_j} u_i(t, |\mathbf{x}|) = \partial_{x_i} u_j(t, |\mathbf{x}|)$  (see (4.7)), we have

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\Omega} \nabla u_i(t, |\mathbf{x}|) \cdot \nabla \psi d\mathbf{x} dt &= \int_{\Omega} \partial_{x_k} u_i(t, |\mathbf{x}|) \partial_{x_k} \psi d\mathbf{x} dt \\ &= \int_{t_1}^{t_2} \int_{\Omega} \partial_{x_i} u_k(t, |\mathbf{x}|) \partial_{x_k} \psi d\mathbf{x} dt \\ &= \int_{t_1}^{t_2} \int_{\Omega} \mathbf{u}_{x_i} \cdot \nabla \psi d\mathbf{x} dt, \end{aligned}$$

whence, by recalling  $\nu = \lambda + 2\mu$ , the last term on the right-hand of (4.15) can be written

$$(\lambda + \mu) \int_{t_1}^{t_2} \int_{\Omega} \mathbf{u}_{x_i} \cdot \nabla \psi d\mathbf{x} dt + \mu \int_{t_1}^{t_2} \int_{\Omega} \nabla u_i \nabla \psi d\mathbf{x} dt.$$

This completes the proof of Proposition 4.5.  $\square$

**Proposition 4.6.** *The weak form (1.19) of the energy equation, as stated in (e) of Theorem 1.1, holds. The total energy  $\mathcal{E}$  of the the limiting functions satisfies (f) of Theorem 1.1.*

PROOF. The proof is the same as that of [9, Propositions 7 and 8], and hence we omit the proof here.  $\square$

Thus, we have completed the proof of Theorem 1.1. In fact, Part (a) of Theorem 1.1 is just Proposition 4.1 with the semicontinuity used implicitly in the proof. The existence and regularity of  $\varrho$ ,  $\mathbf{u}$ ,  $\theta$  asserted in (b) of Theorem 1.1 follow from Propositions 4.2 and 4.3. The weak forms of the mass and momentum equations are proved in Propositions 4.4 and 4.5, from which the regularity assertions in (c) and (d) of Theorem 1.1 follow immediately. Finally, the results in (e) and (f) of the Theorem 1.1 are proved by applying Proposition 4.6.

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